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A NEW SYMBOLISM FOR THE PROPOSITIONAL CALCULUS

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This paper[1] reviews various symbolisms for the two-valued propositional calculus, and introduces a new set of signs which embodies the principles of Leśniewski's symbolism, yet resembles better known signs. This type of symbolism, serving as a diagram, may be used either in place of or as auxiliary to the usual symbolisms.

1. Principal symbolisms. Three symbolisms for the classical propositional calculus are widely used: that of Whitehead and Russell (1910),[2] after Peano, here called the PM symbolism; that of Hilbert and Ackermann (1928); and Łukasiewicz's parenthesis-free symbolism (1929).

In common use are special signs for negation, alternation, conjunction, the conditional, and the biconditional (equivalence), respectively:

$\sim p$, $p \vee q$, $p \cdot q$, $p \supset q$, $p \equiv q$, in PM;

$\overline{X}$, $X \text{ v } Y$, $X \,\&\, Y$, $X \rightarrow Y$, $X \sim Y$, in Hilbert's symbolism;

Np, Apq, Kpq, Cpq, Epq, in Łukasiewicz's symbolism.[3]

We call these five functions "the common truth-functions," the last four "the common binaries."

For the common truth-functions, Hilbert's symbolism agrees with PM only in the sign for alternation (from a different font).[4] PM and Łukasiewicz agree only in the letters for variables.

Next commonest is a sign for dispersion, (non-conjunction), 'not both' —usually Sheffer's stroke, slanting or vertical; 'D' in Łukasiewicz. Several writers have signs for rejection (joint denial), 'neither... nor'.[5] Some use

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[2] Bibliographical references are at the end of the paper.

[3] Many variants of these symbolisms occur. With a PM symbolism, e.g., instead of $\sim p$, p' and $\bar{p}$ are used, the latter in this paper as in Quine (1950). Hilbert and Ackermann (p. 4) give three different signs for the biconditional. Łukasiewicz recently (1951, p. 108) used 'Q' instead of 'E', wanting 'E' for universal negative propositions. Differences in punctuation, especially in the use of dots in a PM symbolism, are often confusing: e.g., '$p \vee q \,.\, r$' is not read by Reichenbach (1948) as by PM.

[4] As if to minimize this sole point of agreement, Hilbert and Ackermann use juxtaposition of variables to express alternation, whereas it often expresses conjunction in PM-based symbolisms.

[5] E.g., the ampheck ⅄ (Peirce, 4.264), inverted wedge $\wedge$ (Sheffer 1913), vertical stroke | (Wittgenstein 1922; used for Boolean algebra by Sheffer 1913), downward arrow $\downarrow$ (Quine 1940).

signs for exjunction (exclusive disjunction), '*aut.*'[6] Signs for other truth-functions are less common.

It is sometimes convenient to have two symbolisms, e.g., one for a language, another for its metalanguage. But to have three principal symbolisms, with many variants and rivals, is a scandal. What is needed, however, is not a fiat, but evaluation of various proposals. No symbolism for the propositional calculus merits universal adoption unless it is complete and systematic.

No variety of the principal symbolisms, to my knowledge, has a special sign for each of the 16 binary truth-functions. Most nearly complete, and most systematic, is Church's (1944) extension of the PM symbolism.[7] This has signs for 10 binaries, viz., the PM signs for the common binaries, also a reverse horseshoe '$\subset$' for the converse conditional 'p if q'; and signs negating each of these by a vertical line through the center. The six other binaries are equivalent to singularies.

2. Peirce's diagonal symbolism. A complete and systematic symbolism for the 16 binaries appears in a Peirce manuscript of 1902 (published 1933). The signs are not printed in the ***Collected papers,*** but the text (4.261) indicates how they are made.

Two intersecting diagonals form equal angles, which, going counter-clockwise from the top, are assigned to the truth-functions $x \cdot y$, $x \cdot \bar{y}$, $\bar{x} \cdot \bar{y}$, $\bar{x} \cdot y$ —written also $\underline{xy}$, $\frac{x}{y}$, $\overline{xy}$, $\frac{y}{x}$ respectively, in Peirce's "ancient system".[8] Then each of the binaries "may be represented by drawing the lines of the diagram between x and y and closing over the compartments for the excluded sets of values." Thus '$x \times y$' represents the tautology; closing over the top represents dispersion, $-(x \cdot y)$; closing over just the left side represents the conditional $-(x \cdot \bar{y})$; closing over top and bottom represents exjunction, $-(x \cdot y) \cdot -(\bar{x} \cdot \bar{y})$; . . . a square with diagonals represents the contradiction.

In this system—Peirce's diagonal system— for *negation* of a function, use the "complementary" sign, i.e., with just those compartments closed over which were originally open; for *conjunction* of functions xFy and xGy, superimpose their symbols; for *alternation*, take the common part of their

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[6] E.g., inverted wedge $\wedge$ (Woodger 1937, Reichenbach 1948), vertical through 3-line $\not\equiv$ (Church 1944), double wedge $\mathbin{\vee\!\!\!\vee}$ (Black 1946).

[7] An analogous extension of Tarski's (1941) mixed symbolism occurs in Gottschalk (1953).

[8] Cf. Peirce, 4.259: "Now it has been the practice, from antiquity, to draw a heavy line under that whose truth it was desired to emphasize. On the other hand, the *obelus*, or spit, is already mentioned by Lucian, in the second century A.D., as the sign of denial;. . ."

symbols. Since symbols with one compartment closed represent the alternations $\bar{x} \vee \bar{y}$, $\bar{x} \vee y$, $x \vee y$, $x \vee \bar{y}$, which conjoined give the other functions (except tautology), the diagonal signs are diagrams of the developed conjunctive normal form for binaries.[9]

3. Leśniewski's wheel symbolism. Stanisław Leśniewski, independent of Peirce, developed a complete and systematic symbolism for the 16 binaries,[10] here called the "wheel symbolism." The signs consist of a small circle (the hub), from which zero to four straight lines (the spokes) radiate in vertical and/or horizontal direction. '⫯ (pq)' stands for '$p \,.\, q$', conjunction; '⊸ (pq)' for '$p \,.\, \bar{q}$', sinejunction; '⫰ (pq)' for '$\bar{p} \,.\, \bar{q}$', rejection; '⟜ (pq)' for '$\bar{p} \,.\, q$', subjunction. The sign for each binary has just the spokes corresponding to alternatives in its Boolean expansion (developed alternative normal form). E.g., the sign for the conditional has all but the left spoke.[11] '○ (pq)' stands for the contradiction, the (rimless) wheel for the tautology.

Rules for this system correspond by duality to those for Peirce's. The fundamental principle follows from the construction of the signs: (1) The sign for each binary is a diagram of its *Boolean expansion.*

Rules (2)–(5) are corollaries. (2) For *negation* of a function, use the "complementary" sign (with just the spokes not in the original). (3) For *alternation* of functions $F(pq)$ and $G(pq)$, superimpose their symbols. (4) For *conjunction* of $F(pq)$ and $G(pq)$, take the common part of their symbols. (5) 'If $F(pq)$ then $G(pq)$' is a tautology if and only if the sign for F is part or whole of that for G.

Similar rules are easily devised (e.g., for forming a conditional, judging compatibility, etc.). In addition, certain rules arise from the symmetry of the signs: (6) For *conversion* of a binary, use its sign reversed, i.e., the mirror image, with variables in original order. (E.g., the sign for the converse conditional is the mirror image of that for the conditional.) Commutative functions alone have signs with bilateral symmetry. (7) Negating all variables of a truth-function (e.g., $p \supset q$) gives a function ($\bar{p} \supset \bar{q}$) I call the *intraverse* (or *logical inverse*) of the original. For intraversion of a binary, "invert" its sign, i.e., rotate it 180 degrees. (8) Since the *dual* of a truth-

[9] The points of this paragraph are not in Peirce. The diagonal signs could be interpreted as diagrams of a developed *alternative* normal form, by considering the compartments left open rather than those closed over.

[10] Leśniewski (1929, etc.) used this symbolism to a limited extent. It may be found in a recent monograph by Sobociński (1949), but without the analysis of its characteristics given here.

[11] This sign for the conditional is the only one of the wheel signs for which I see a mnemonically helpful resemblance to known signs, pointed out by Y. T. Shen some twenty years ago. Resembling the assertion sign, it suggests, if written between ist arguments, the now common '$p \vdash q$', "p yields q."

function is equivalent to the negation of its intraverse,[12] the dual of $F(pq)$ uses the sign complementary to that for its intraverse, (equivalently, inverts the complementary sign). (E.g., the dual of a three-spoke binary uses the spoke "in the middle.")

No such simple rules hold for the principal symbolisms, except that rule 6 holds in Church's extension of PM symbolism, and a simple rule for negation. Any problems in two variables solvable by truth-tables can be solved by inspection or manipulation of Peircian or Leśniewskian signs.

Leśniewski's symbolism—its advantages not publicized—has been little used. This may be mainly due to typographical difficulty. But if typographical facility were decisive, Łukasiewicz's symbolism should have supplanted all others. Psychological factors must also be considered, especially similarity and analogy to—and avoidance of conflict with—known symbols. Ease of writing and graphic distinctiveness also count. The best symbolism, we contend, will have these qualities while obeying rules like the Leśniewskian rules above.

4. The new symbolism. We take our signs for the four alternants in the Boolean expansion of a binary tautology from the sides of a quadrilateral. This symbolism is presented in two variants: the trapezoid symbolism and the square symbolism.

The trapezoid symbolism is the form previously expounded,[13] and more convenient for handwriting and an ordinary typewriter. Neither form is readily printed with existing type, but the square symbolism might be easier to do and look neater. The trapezoid symbolism will be explained, and the distinctive points of the square symbolism.

The **trapezoid symbolism** takes its signs from an isosceles trapezoid, with top ($\frac{1}{2}$ to 3 times) longer than base. We put ‘$p \,\_\, q$’ for $p \cdot q$, ‘$p\backslash\ \ q$’ for $p \cdot \bar{q}$, ‘$p \overline{\quad} q$’ for $\bar{p} \cdot \bar{q}$, ‘$p\ \ | q$’ for $\bar{p} \cdot q$. The sign for each binary has just the sides representing the alternants in its Boolean expansion. E.g., ‘$p\backslash\!\_\, q$’ stands for $(p \cdot q) \vee (p \cdot \bar{q})$, ‘$p \square q$’ for the tautology.[14]

This symbolism, unlike Leśniewski's, does not put a sign for the contradiction with every sign. The contradiction as a function is represented by a cross or ‘x’: ‘$p \times q$’ for $p \cdot \bar{p} \cdot q$.[15]

The eight Leśniewskian rules (§ 3) hold, with sides of the trapezoid re-

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[12] Cf. Quine (1950), p. 61, "second law of duality."

[13] Cf. footnote 1; I have also taught this for six years at the University of Buffalo.

[14] For binaries, we write the quadrilateral signs between their arguments. Łukasiewicz's parenthesis-free style would conflict with the proposal below (§ 6) for n-ary functions.

[15] If a strict one-to-one correspondence with Leśniewski's symbolism were wanted, the ‘x’ would be put with every sign. The sides of a square could then be used without modification.

placing the respective spokes of the wheel; except that the contradiction is treated as if its sign were a blank space, 'x' replacing a final blank; and except that for intraversion (rule 7), in addition to inverting the sign, the length or slant of the sides must be adjusted.[16] Typographical difficulties aside, the existence of exceptions to the rules is the principal disadvantage of the trapezoid compared with the wheel.

The trapezoid symbolism has definite advantages over the wheel, while maintaining practically all advantages of the latter. Most important, several of our signs suggest known signs, which may be thought of as variants.

(i) The conjunction-line of '$p \_ q$' may be shortened to a dot, giving '$p \,.\, q$' as an "abbreviation." (Or we may write the conjunction-line as an underline, '$\underline{pq}$', as in the ancient system (§ 2).)

(ii) '$p^{\overline{\ \ }}q$', "neither p nor q," may be written '$\overline{p\ q}$' or '$\overline{pq}$', which suggest the PM variants '$\bar{p} \,.\, \bar{q}$' or '$\bar{p}\bar{q}$'. (The ancient system also indicates rejection by a bar over the arguments.)[17]

(iii) The conditional sign '¯|' suggests PM's horseshoe '⊃', which may be regarded as a cursive or "italic" form of the same sign.

(iv) For the converse conditional, '|¯' suggests the reversed horseshoe '⊂' (of Church).

(v) For alternation, '_/' suggests the wedge or 'v', and the cup '◡' of the class calculus.

(vi) For exjunction, '\ /' may be thought of as a broken wedge, suggesting an exclusive *or*.

(vii) The biconditional sign '¯_', representing equality of truth-value, is a modified equality sign.

The trapezoid system is designed to give the maximum resemblance to known signs, especially for the common binaries (odd numbers above), while following Leśniewskian rules. Its signs can be written more quickly than those of the wheel or square systems; none of the systems require exact drawing. Also, except for the left side, the trapezoidal signs can be typed on an ordinary typewriter. (The top is made of three underlines.) The trapezoidal signs, being asymmetrical about the *horizontal* axis, have a slight advantage in graphic distinctiveness.

The **square symbolism** takes its signs from a square. Going clockwise from the bottom, the sides are used for $p \,.\, q$, $p \,.\, \bar{q}$, $\bar{p} \,.\, \bar{q}$, $\bar{p} \,.\, q$, respectively. To distinguish the left and right sides as separate signs, we use brackets: '$p[\ q$' for $p \,.\, \bar{q}$, '$p\ ]q$' for $\bar{p} \,.\, q$. (Position is usually sufficient to distinguish top and bottom; if another distinction is wanted, put '$p^{\overline{\ \ }}q$' for $\bar{p} \,.\, \bar{q}$.)

[16] The writer (like Gottschalk) likes to treat of intraversion along with duality and negation, but finds no difficulty in making the necessary adjustments.

[17] The accidental coincidence with Hilbert's '$\overline{XY}$' (for '$\overline{X \vee Y}$') is not helpful. The use of a negation-bar over compound expressions would be confusing in a quadrilateral symbolism.

The Leśniewskian rules hold, *mutatis mutandis*, except for contradiction again, represented by 'x', and except that the sides are treated as straight lines, a bracket replacing a vertical side left without an adjacent side.

The resemblances and variants set out for the trapezoidal signs hold for the square signs, except that the square signs for alternation and exjunction do not suggest the wedge. The square signs for (iii), (iv), (vii) are closer to the variants than the trapezoidal signs.[18]

5. Singulary functions. Negation is the only singulary truth-function for which a sign is ordinarily used, but signs for the others are occasionally desirable. In Leśniewski's system of singulary signs, a long horizontal line represents the contradiction ('$—p$' for $p \,.\, \bar{p}$), and appears in all the singulary signs. To this, add a short vertical line bisected by the left {right} end, for negation {affirmation}; or such lines at both ends for the tautology.[19] The rules for binaries, except that for conversion, hold, *mutatis mutandis*, for Leśniewski's singularies.

In devising singulary signs for the trapezoid system, we note that p‾⌉q reduces to $\bar{p}$, and p_q to p. Rectifying the angles, we take '⌉p' for negation and '⌊p' for affirmation. Their common part is the vertical line, so ' |p' stands for the contradiction; and the signs may be combined to give '⌉⌊p' for the tautology.[20]

This negation sign occurs in Heyting's intuitionist logic;[21] but I prefer to write it in "abbreviated" form, omitting the vertical line. This gives the familiar short dash or bar, which may be written before or (in this system, for variables only) above its argument.[22]

6. *n*-ary functions. Generalization to a symbolism for n variables offers no difficulty in principle. It requires a sign with 2^n distinguishable parts, corresponding to the alternants in the Boolean expansion of an n-ary tautology, (with another element for the contradiction). But the problem is to get a practical symbolism that fits in with the binary symbol-

[18] Other variant forms of quadrilateral symbolism have their points. The vertical sides might be parentheses. A trapezoid with base longer than top would give good signs for sinejunction and subjunction, as Mr. Carl Stanton points out.

[19] If Leśniewski had interchanged his seldom used signs for affirmation and negation, the sign for affirmation would have resembled PM's assertion sign (with vertical line shorter).

[20] For the square system, the square signs for $\bar{p}$ and p should be modified, e.g., by halving the horizontal lines, to give singulary signs.

[21] The use of Heyting's negation sign for the classical calculus is rare, but occurs, e.g., in Curry (1950).

[22] Omitting the vertical line from the affirmation sign gives a sign like Frege's horizontal line. This is recommended only when the horizontal line is written as an underline, coinciding with the underline in the ancient system.

ism. For example, the sign for each binary function of p, q should be simply related to the sign for the equivalent degenerate function of p, q, r.[23]

Mr. Gerald B. Standley has solved this problem for any Leśniewskian system: to obtain the 2^n parts ($n > 1$), write the binary tautology sign 2^{n-2} times. Where, e.g., F, G are binary functors of the system, $FGpqr$ means $(pFq) \,.\, r \,.\vee.\, (pGq) \,.\, \bar{r}$. Except the rule for conversion,[24] the rules for Leśniewskian binaries hold for n-ary signs constructed in this way, with the qualifications noted in § 4 for quadrilateral signs. Even with a non-Leśniewskian symbolism, a Leśniewskian symbolism as generalized by Standley is a useful auxiliary for problem-solving.[25]

Note on Terminology

We prefer simple names for the truth-functions: thus "dispersion" rather than "alternative denial" or "non-conjunction." (A "non-conjunctive proposition," in the standard use of "non-", would be any proposition not conjunctive. "Incompatibility," having modal connotation, is inappropriate here.) "Dispersion," "rejection," and "exjunction" come from Sheffer's writings or lectures.

"Sinejunction" and "subjunction" I believe are new. "Sinejunction" is suggested (i) by the use of *sans* for this or analogous functions in French, and (ii) by analogy with "*con*junction," i.e., "with-junction." ("Exception" has been used for an analogous modal function S by Moisil, reviewed XIII 162, but seems more appropriate to the class calculus.) "Subjunction" is suggested by (i) the symbol $\frac{y}{x}$ for $\bar{x} \,.\, y$ in the ancient system (§ 2), and (ii) the term "subsection" for this function (and its analogue in the class calculus) in Boll and Reinhart (1946).

"Intraverse" is introduced (§ 3) for Gottschalk's (1953) acceptable term "contradual," because (i) the former term better suggests the operation of negating variables within a function; (ii) though the intraverse is equivalent to the *contra*dictory of the *dual* in the propositional calculus and other systems, this equivalence fails in some logical systems: the intraverse of "All S are P" is "All non-S are non-P," but the contradictory of its dual is "No S are P"; and (iii) "intraversion" suggests "inversion," appropriate because (α) this operation is expressed by *inverting* the sign in Leśniewskian symbolisms, or *inverting* the principal column in a truth-table (Gottschalk), and (β) "if not p then not q" is called the inverse of "if p then q," and an analogous use is found for categorical propositions. (I no longer call this operation "inversion," but might call it "logical inversion" as distinct from mathematical.)

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[23] The best way of representing ternaries I could find was to divide each side of a quadrilateral into an r half and a $\bar{r}$ half. This is awkward and cannot be extended.

[24] The rule for the converse might be extended to n variables in this way: Interchanging the first two variables of a function is equivalent to reversing each functor-sign.

[25] Before learning of Leśniewskian symbolisms, Mr. Standley developed a somewhat similar symbolism in 1952 simply as short-cut for truth-tables.

Inter alia, a generalized quadrilateral symbolism facilitates solution of Quine's (1952) problem of simplifying truth functions. Merging the signs for a function gives a diagram of the Boolean expansion, from which (by noting repetitions and angles) the "prime implicants" can be picked out by inspection.

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